Introduction to Quantum Physics

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Abstract

The video channel with the title "Professor M does Science" in YouTube offers a simple step-by-step but all the same very valuable and rigorous introduction into the world of quantum physics. This script covers the harmonic oscillator and helps to digest the topic covered by a group of those videos but is not meant as a replacement for them.

5 The Harmonic Oscillator and its Coherent States

5.1 Classical and Quantum Harmonic Oscillators

Applications of the harmonic oscillator in classical physics ranges from the vibrations of strings to the behavior of electronic circuits. In the quantum world the harmonic oscillator allows to describe the motion of atoms in solids and the behavior of light to list just two examples. The quantum harmonic oscillator is also one of the few problems in quantum physics one knows how to solve exactly.

The defining feature of the harmonic oscillator is that it is a system with a potential energy that depends quadratically on the amplitude. Its potential energy is usually written as $V(x) = \frac{1}{2} k\, x^2$ with $k > 0$. In classical mechanics the force experienced by a particle is given by the negative of the gradient of the potential such that it is

$$F = -\frac{d}{dx} V(x) \qquad\qquad F = \frac{d}{dx}\left(\frac{1}{2} k\, x^2\right) = -k\, x$$

for potentials in general on the left side and for the harmonic oscillator specifically on the right side. The equation $F = -k\, x$ is called Hooke's law. Its solution in the form

$$V(x) = \frac{1}{2} m\, \omega^2\, x^2 \qquad\qquad \omega = \sqrt{\frac{k}{m}}$$

is a harmonic motion of frequency ω.

In a general potential $V(x)$ with a minimum at x_0, a particle at x_0 does not move because

$$F = -\frac{d}{dx} V(x)\Big|_{x_0} = 0$$

and the potential in the environment of x_0 can be approximated using a Taylor expansion

$$V(x) = V(x_0) + \frac{dV(x)}{dx}\Big|_{x_0}(x - x_0) + \frac{1}{2}\frac{d^2V(x)}{dx^2}\Big|_{x_0}(x - x_0)^2 + \frac{1}{6}\frac{d^3V(x)}{dx^3}\Big|_{x_0}(x - x_0)^3 + ...$$

in the environment of x_0. Because the potential is expanded about the minimum x_0 the first order term $\frac{dV(x)}{dx}$ vanishes. For small enough displacements the dominant term in the expansion will be the lowest order term. Thus, the potential can be approximated

$$V(x) \approx V(x_0) + \frac{1}{2}\frac{d^2V(x)}{dx^2}\Big|_{x_0}(x - x_0)^2$$

close to the minimum x_0. This is one reason why the harmonic oscillator is so important. Whatever arbitrary potential is given the one-dimensional motion of a particle close enough to a local minimum can be approximated by this quadratic equation where the $V(x_0)$ and the x_0 in $(x-x_0)^2$ can be shifted away. This approach can be generalized to higher dimensions

$$V(x_1,...,x_N) \approx \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial^2 V}{\partial x_j\,\partial x_k}\Big|_0 x_j\,x_k \qquad\qquad V(y_1,...,y_N) \approx \frac{1}{2}\sum_{j=1}^{N}\frac{\partial^2 V}{\partial y_j^2}\Big|_0 y_j^2$$

where the potential and the coordinates already have been shifted on the left side and where the coordinates have been transformed on the right side to eliminate the cross-terms. This transformation is always possible for Hamiltonians with a quadratic potential. Thus, the Hamiltonian can be written in a form with N decoupled harmonic oscillators. These coordinates y_j are called *normal modes*. To move to the harmonic oscillator in quantum mechanics the classical total energy E has to be promoted to operators

$$E = \frac{p^2}{2m} + \frac{1}{2}\,m\,\omega^2\,x^2 \qquad\qquad \hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}\,m\,\omega^2\,\hat{x}^2$$

as usual for the quantization of classical systems.

5.2 Ladder and Number Operators for the Quantum Harmonic Oscillator

Similarly to the corresponding ladder operators for angular momentum, the lowering operator $\hat{a}$ and the raising operator $\hat{a}^\dagger$ for the harmonic oscillator can be defined as the *ladder operators*

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\,\omega}{\hbar}}\,\hat{x} + i\frac{1}{\sqrt{m\,\hbar\,\omega}}\,\hat{p}\right) \qquad\qquad \hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\,\omega}{\hbar}}\,\hat{x} - i\frac{1}{\sqrt{m\,\hbar\,\omega}}\,\hat{p}\right) \tag{5.1}$$

in terms of the position operator $\hat{x}$ and the momentum operator $\hat{p}$ satisfying the commutation relation $[\hat{x},\hat{p}] = i\,\hbar$. The position and momentum operator can be written as

$$\hat{x} = \sqrt{\frac{\hbar}{2\,m\,\omega}}\left(\hat{a}^\dagger + \hat{a}\right) \qquad\qquad \hat{p} = i\sqrt{\frac{m\,\hbar\,\omega}{2}}\left(\hat{a}^\dagger - \hat{a}\right)$$

using these ladder operators. The two operators $\hat{a}$ and $\hat{a}^\dagger$ are not observables because they are not Hermitian and are each others adjoint. The only non-zero commutation relation is $[\hat{a},\hat{a}^\dagger] = 1$ because

$$\begin{aligned}\left[\hat{a},\hat{a}^\dagger\right] &= \left[\frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\,\omega}{\hbar}}\,\hat{x} + i\frac{1}{\sqrt{m\,\hbar\,\omega}}\,\hat{p}\right), \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\,\omega}{\hbar}}\,\hat{x} - i\frac{1}{\sqrt{m\,\hbar\,\omega}}\,\hat{p}\right)\right] \\ &= \frac{1}{2}\left(\frac{m\omega}{\hbar}\,[\hat{x},\hat{x}] + \frac{1}{m\hbar\omega}\,[\hat{p},\hat{p}] - \frac{i}{\hbar}\,[\hat{x},\hat{p}] + \frac{i}{\hbar}\,[\hat{p},\hat{x}]\right) = \frac{1}{2}\left(-\frac{i}{\hbar}(i\hbar) + \frac{i}{\hbar}(-i\hbar)\right) = 1\end{aligned}$$

using the commutation relations of $\hat{x}$ and $\hat{p}$.

The *number operator* $\hat{N}$ is defined as $\hat{N} = \hat{a}^\dagger\hat{a}$. Unlike the two ladder operators the number operator is Hermitian because $\hat{N}^\dagger = (\hat{a}^\dagger\hat{a})^\dagger = \hat{a}^\dagger(\hat{a}^\dagger)^\dagger = \hat{a}^\dagger\hat{a} = \hat{N}$. All the relevant commutation relations between the ladder operators and the number operator are

$$\left[\hat{a},\hat{a}^\dagger\right] = 1 \qquad\qquad \left[\hat{N},\hat{a}\right] = -\hat{a} \qquad\qquad \left[\hat{N},\hat{a}^\dagger\right] = \hat{a}^\dagger$$

as $[\hat{N},\hat{a}] = [\hat{a}^\dagger\hat{a},\hat{a}] = \hat{a}^\dagger[\hat{a},\hat{a}] + [\hat{a}^\dagger,\hat{a}]\hat{a} = 0 - \hat{a} = -\hat{a}$ shows using $[\hat{A},\hat{B}\hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A}\hat{C}]\hat{B}$.

If $\hat{N}\,|\lambda\rangle = \lambda\,|\lambda\rangle$ is the eigenvalue equation of the number operator with the eigenstate $|\lambda\rangle$ then $\hat{a}\,|\lambda\rangle$ is also an eigenstate of $\hat{N}$ with eigenvalue $\lambda - 1$ as

$$\hat{N}\,\hat{a}\,|\lambda\rangle = \hat{a}\hat{N}\,|\lambda\rangle - \hat{a}\,|\lambda\rangle = \lambda\,\hat{a}\,|\lambda\rangle - \hat{a}\,|\lambda\rangle = (\lambda-1)\,\hat{a}\,|\lambda\rangle$$

proves using the commutator $[\hat{N},\hat{a}] = -\hat{a}$ in the form $\hat{N}\hat{a} = \hat{a}\hat{N} - \hat{a}$. This further means that one can write $\hat{a}\,|\lambda\rangle = c_-\,|\lambda-1\rangle$ such that $\hat{a}$ applied to an eigenstate $|\lambda\rangle$ of $\hat{N}$ gives another eigenstate $|\lambda-1\rangle$ of $\hat{N}$. This is the reason for the name lowering operator. Repeating these calculations for $\hat{a}^\dagger$ shows with

$\hat{N}\,|\lambda\rangle = \lambda\,|\lambda\rangle$ that $\hat{a}^\dagger\,|\lambda\rangle$ is also an eigenstate of $\hat{N}$ with eigenvalue $\lambda+1$. It also follows $\hat{a}^\dagger\,|\lambda\rangle = c_+\,|\lambda+1\rangle$ showing the reason for calling it a raising operator.

To determine c_- and c_+ in $\hat{a}\,|\lambda\rangle = c_-\,|\lambda-1\rangle$ and $\hat{a}^\dagger\,|\lambda\rangle = c_+\,|\lambda+1\rangle$ one can calculate

$$\|\hat{a}\,|\lambda\rangle\|^2 = \langle\lambda|\hat{a}^\dagger\hat{a}|\lambda\rangle = \langle\lambda|\hat{N}|\lambda\rangle = \lambda\,\langle\lambda|\lambda\rangle = \lambda \qquad \|c_-\,|\lambda-1\rangle\|^2 = |c_-|^2\,\langle\lambda-1|\lambda-1\rangle = |c_-|^2$$

such that c_- can be chosen as $\sqrt{\lambda}\in\mathbb{R}$. This together with similar calculations for $\hat{a}^\dagger$ shows that

$$\hat{a}\,|\lambda\rangle = \sqrt{\lambda}\,|\lambda-1\rangle \qquad \hat{a}^\dagger\,|\lambda\rangle = \sqrt{\lambda+1}\,|\lambda+1\rangle \tag{5.2}$$

are the actions of the ladder operators on eigenstates of $\hat{N}$.

Using (5.1) to determine the number operator $\hat{N}$ gives

$$\begin{aligned}\hat{N} = \hat{a}^\dagger\hat{a} &= \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\,\omega}{\hbar}}\,\hat{x} - i\frac{1}{\sqrt{m\,\hbar\,\omega}}\,\hat{p}\right)\frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\,\omega}{\hbar}}\,\hat{x} + i\frac{1}{\sqrt{m\,\hbar\,\omega}}\,\hat{p}\right)\\ &= \frac{1}{2}\left(\frac{m\,\omega}{\hbar}\,\hat{x}^2 + \frac{1}{m\,\hbar\,\omega}\,\hat{p}^2 + \frac{i}{\hbar}\,\hat{x}\,\hat{p} - \frac{i}{\hbar}\,\hat{p}\,\hat{x}\right) = \frac{1}{2}\left(\frac{m\,\omega}{\hbar}\,\hat{x}^2 + \frac{1}{m\,\hbar\,\omega}\,\hat{p}^2 + \frac{i}{\hbar}\,[\hat{x},\hat{p}]\right)\\ &= \frac{1}{2}\left(\frac{m\,\omega}{\hbar}\,\hat{x}^2 + \frac{1}{m\,\hbar\,\omega}\,\hat{p}^2 - 1\right)\end{aligned}$$

and

$$\hbar\,\omega\left(\hat{N}+\frac{1}{2}\right) = \hbar\,\omega\left(\frac{m\,\omega}{2\,\hbar}\,\hat{x}^2 + \frac{1}{2\,m\,\hbar\,\omega}\,\hat{p}^2\right) = \frac{1}{2\,m}\hat{p}^2 + \frac{1}{2}\,m\,\omega^2\,\hat{x}^2 = \hat{H}$$

such that

$$\hat{H} = \frac{1}{2\,m}\hat{p}^2 + \frac{1}{2}\,m\,\omega^2\,\hat{x}^2 = \hbar\,\omega\left(\hat{N}+\frac{1}{2}\right) = \hbar\,\omega\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right) \tag{5.3}$$

follows for the Hamiltonian.

5.3 Energy Eigenvalues of the Quantum Harmonic Oscillator

The energy eigenvalues E_λ are the solutions of the eigenvalue equation $\hat{H}\,|\lambda\rangle = E_\lambda\,|\lambda\rangle$ for the Hamiltonian (5.3). These values are real numbers because $\hat{H}$ is Hermitian. The calculation

$$\hat{H}\,|\lambda\rangle = \hbar\,\omega\left(\hat{N}+\frac{1}{2}\right)|\lambda\rangle = E_\lambda\,|\lambda\rangle \quad\Rightarrow\quad \hat{N}\,|\lambda\rangle = \left(\frac{E_\lambda}{\hbar\,\omega} - \frac{1}{2}\right)|\lambda\rangle = \lambda\,|\lambda\rangle \quad\Rightarrow\quad \lambda = \frac{E_\lambda}{\hbar\,\omega} - \frac{1}{2}$$

shows that $|\lambda\rangle$ is an eigenstate of $\hat{H}$ and $\hat{N}$ but the eigenvalues are different. The eigenvalue of $\hat{H}$ is E_λ, the eigenvalue of $\hat{N}$ is λ and

$$\lambda = \frac{E_\lambda}{\hbar\,\omega} - \frac{1}{2} \qquad E_\lambda = \hbar\,\omega\left(\lambda + \frac{1}{2}\right)$$

is the relation between the two eigenvalues.

So far, the only known property of λ is that it is real because $\hat{N}$ is Hermitian. With

$$\langle\lambda|\hat{N}|\lambda\rangle = \lambda\,\langle\lambda|\lambda\rangle = \lambda \qquad \langle\lambda|\hat{N}|\lambda\rangle = \langle\lambda|\hat{a}^\dagger\hat{a}|\lambda\rangle = \|\hat{a}\,|\lambda\rangle\|^2 \geq 0$$

one finds as the next property that $\lambda \geq 0$. From (5.2) with $\hat{a}\,|\lambda\rangle = \sqrt{\lambda}\,|\lambda-1\rangle$ one concludes $\hat{a}\,|\lambda\rangle = 0$ if $\lambda = 0$. It further follows $\lambda\in\mathbb{N}$ because otherwise one can apply $\hat{a}$ in steps of -1 until one reaches a λ between 0 and 1 such that one cannot apply $\hat{a}$ again due to the constraint $\lambda \geq 0$. In other words, λ_{min} must exist and must be the value 0 for which $\hat{a}\,|\lambda\rangle = 0$, and therefore $\lambda_{\text{min}} = 0$. Because λ must be a non-negative integer, the eigenvalue equation for $\hat{N}$ is usually written as $\hat{N}\,|n\rangle = n\,|n\rangle$ for $n = 0, 1, 2, \ldots$.

Consequently, the eigenvalue equation for $\hat{H}$ is written as $\hat{H}\,|n\rangle = E_n\,|n\rangle$, and

$$E_n = \hbar\,\omega\left(n + \frac{1}{2}\right) \tag{5.4}$$

are the energy eigenvalues where $0, 1, 2, 3, ...$ are the possible values for n. Therefore, the energy values for the quantum harmonic oscillator are quantized. The lowest possible value is $\frac{1}{2}\hbar\omega$, the next is $\frac{3}{2}\hbar\omega$ and so on. Two neighboring values are separated by $\hbar\omega$. The fact that the lowest possible energy value is not zero means that such a quantum particle can never be completely at rest. The value $\hbar\omega$ is called the *quantum of energy*, and E_0 is called the zero point energy.

The lowering operator $\hat{a}$ leads from energy eigenstate $|n\rangle$ to energy eigenstate $|n-1\rangle$ and removes $\hbar\omega$ energy. The raising operator $\hat{a}^\dagger$ leads from energy eigenstate $|n\rangle$ to energy eigenstate $|n+1\rangle$ and increases the energy by $\hbar\omega$. The number operator $\hat{N}$ gives the number of energy quanta when applied to an energy eigenstate, and this is obviously the reason why it is called the number operator.

5.4 Energy Eigenstates of the Quantum Harmonic Oscillator

The raising operator $\hat{a}^\dagger$ acts as

$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad \hat{a}^\dagger\,|n-1\rangle = \sqrt{n}\,|n\rangle \qquad |n\rangle = \frac{1}{\sqrt{n}}\,\hat{a}^\dagger\,|n-1\rangle = \frac{1}{\sqrt{n(n-1)}}(\hat{a}^\dagger)^2\,|n-2\rangle = ...$$

on the energy eigenstates $|n\rangle$ such that $|n\rangle$ can be written as

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n\,|0\rangle \tag{5.5}$$

showing that the application of $\hat{a}^\dagger$ on the ground state n times gives the n^{th} energy eigenstate. The factor $1/\sqrt{n!}$ ensures that $|n\rangle$ is normalized if $|0\rangle$ was normalized.

The next goal is to write the energy eigenstates $|n\rangle$ as wave functions in the position representation. The lowering operator acting on the ground state

$$\hat{a}\,|0\rangle = 0 \qquad\qquad \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\,\omega}{\hbar}}\,\hat{x} + i\frac{1}{\sqrt{m\,\hbar\,\omega}}\,\hat{p}\right)|0\rangle = 0$$

from (5.1) corresponds to

$$\left(\frac{m\,\omega}{\hbar}\,x + \frac{d}{dx}\right)\psi_0(x) = 0 \qquad\qquad \psi_0(x) = \langle x|0\rangle$$

in the position representation. As a first-order differential equation one can rearrange it to

$$\frac{1}{\psi_0(x)}\frac{d\psi_0(x)}{dx} = -\frac{m\,\omega}{\hbar}x \qquad\qquad \int\frac{1}{\psi_0(x)}\frac{d\psi_0(x)}{dx}\,dx = -\frac{m\,\omega}{\hbar}\int x\,dx$$

such that one can apply separation of variables and integration. This gives

$$\ln\big(\psi_0(x)\big) = -\frac{m\,\omega}{2\hbar}x^2 + c \qquad\qquad \psi_0(x) = A\,e^{-m\,\omega\,x^2/2\hbar}$$

and shows that the ground state wave function $\psi_0(x)$ is a Gaussian satisfying

$$\int_{-\infty}^{\infty} e^{-\alpha\,x^2}\,dx = \sqrt{\frac{\pi}{\alpha}} \qquad\qquad \alpha = \frac{m\,\omega}{\hbar}$$

and

$$1 = \int_{-\infty}^{\infty}|\psi_0(x)|^2\,dx = |A|^2\sqrt{\frac{\pi\,\hbar}{m\,\omega}} \qquad\qquad \psi_0(x) = \left(\frac{m\,\omega}{\pi\,\hbar}\right)^{1/4} e^{-m\,\omega\,x^2/2\hbar}$$

where a phase choice ensured that A is real. Using

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n\,|0\rangle \qquad\qquad \hat{a}^\dagger = \sqrt{\frac{m\,\omega}{2\hbar}}x - \sqrt{\frac{\hbar}{2m\omega}}\frac{d}{dx}$$

one can calculate $\psi_n(x)$ with

$$\psi_n(x) = \frac{1}{\sqrt{n!}}\left(\sqrt{\frac{m\,\omega}{2\hbar}} - \sqrt{\frac{\hbar}{2m\omega}}\frac{d}{dx}\right)^n \psi_0(x) = \frac{1}{\sqrt{2^n\,n!}}\left(\frac{m\,\omega}{\pi\,\hbar}\right)^{1/4}\left(\sqrt{\frac{m\,\omega}{\hbar}} - \sqrt{\frac{\hbar}{m\omega}}\frac{d}{dx}\right)^n e^{-m\,\omega\,x^2/2\hbar}$$

from the wave function of the ground state. As an example

$$\psi_1(x) = \sqrt{\frac{2\,m\,\omega}{\hbar}}\left(\frac{m\,\omega}{\pi\,\hbar}\right)^{1/4} x\,e^{-m\,\omega\,x^2/2\hbar}$$

is the first excited state. Each excited state $\psi_n(x)$ has the general form of a prefactor times a polynomial of order n in x times the Gaussian exponential $e^{-m\,\omega\,x^2/2\hbar}$. The solutions are usually written in the form

$$\psi_n(x) = \frac{1}{\sqrt{2^n\,n!}}\left(\frac{m\,\omega}{\pi\,\hbar}\right)^{1/4} H_n\left(\sqrt{\frac{m\,\omega}{\hbar}}x\right) e^{-m\,\omega\,x^2/2\hbar}$$

where $H_n(z)$ are the Hermite polynomials with definite parity $H_n(-z) = (-1)^n H_n(z)$. Because the Gaussian is an even function, the energy eigenfunctions satisfy $\psi_n(-x) = (-1)^n\psi_n(x)$ and have the same parity as that of the Hermite polynomials.

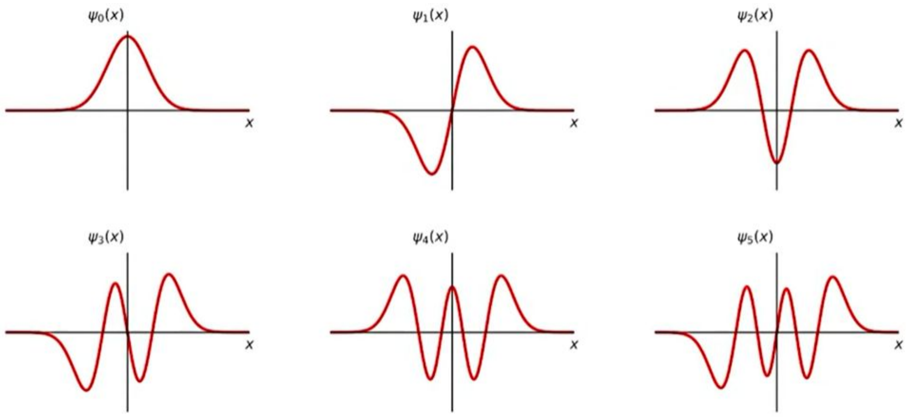
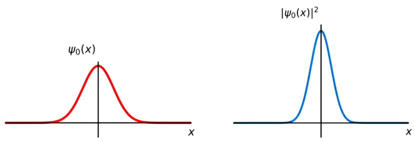
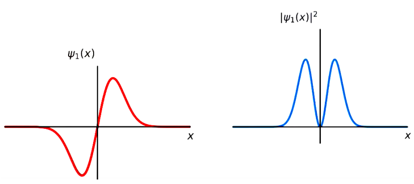


Figure 1: The first six eigenfunctions for the harmonic oscillator

The first six eigenfunctions of the quantum harmonic oscillator are shown in figure 1 where the first one is the ground state $\psi_0(x)$ and the others the first five excitations. The corresponding probability distributions are presented in figure 2 below. The relation between eigenfunction and probability distribution is discussed for the ground state $\psi_0(x)$ and the first excited state $\psi_1(x)$.

The eigenfunction $\psi_0(x)$ for the ground state shown as the left graph in the figure on the right side is proportional to a Gaussian function and is therefore an even function. The corresponding probability distribution $|\psi_0(x)|^2$ in the right graph illustrates that the particle is with high probability at the center.





The first excited state wave function $\psi_1(x)$ presented as the left graph in the figure on the left side is an odd function because it is x times a Gaussian function. The right graph in this figure shows that the probability to find the particle at the center is zero, and the highest probabilities are located at the two peaks symmetrically located on both sides of the vertical axis through the center.

In general, the eigenfunctions $\psi_n(x)$ are odd functions if n is odd and are even functions if n is even. The probability densities $|\psi_n(x)|^2$ are always even functions as figure 2 shows for the first six eigenfunctions. For odd n the probability to find the particle at the center is zero.

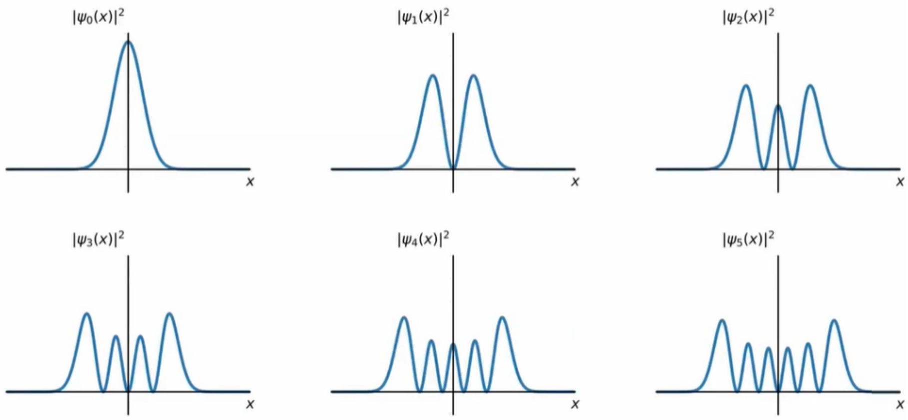
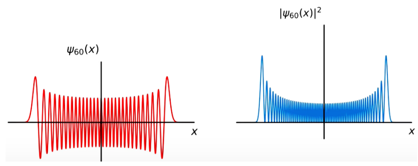


Figure 2: The first six probability distributions for the harmonic oscillator



The number of x values with $\psi_n(x) = 0$ and the number of peaks of the probability densities $|\psi_n(x)|^2$ increase with n. The case for $n = 60$ in the figure on the right side illustrates this, and it also shows that the probability to find the particle is highest at the two peaks farthest away from the center. This result is consistent with the classical limit because these peaks correspond to the turning points of the classical harmonic oscillator where it is momentarily at rest while the center is the location where it moves fastest.

5.5 The Translation Operator

The position operator $\hat{x}$ and the momentum operator $\hat{p}$ do not commute as the well-known case of the Heisenberg uncertainty principle shows, and their commutation relations are $[\hat{x}, \hat{p}] = i\,\hbar$. The translation operator already introduced in an earlier chapter but needed here is defined as

$$\hat{T}(\alpha) = e^{-i\,\alpha\,\hat{p}/\hbar} \qquad\qquad \alpha \in \mathbb{R} \tag{5.6}$$

and it translates in space by an amount α. The adjoint operator is

$$\hat{T}^\dagger(\alpha) = e^{i\,\alpha\,\hat{p}^\dagger/\hbar} = e^{i\,\alpha\,\hat{p}/\hbar} = e^{-i\,(-\alpha)\,\hat{p}/\hbar} = \hat{T}(-\alpha)$$

using $\hat{p}^\dagger = \hat{p}$. The translation operator is therefore not Hermitian but is unitary as

$$\hat{T}^\dagger(\alpha)\hat{T}(\alpha) = e^{i\,\alpha\,\hat{p}/\hbar}\,e^{-i\,\alpha\,\hat{p}/\hbar} = \mathbb{I} \qquad\qquad \hat{T}(\alpha)\hat{T}^\dagger(\alpha) = e^{-i\,\alpha\,\hat{p}/\hbar}\,e^{i\,\alpha\,\hat{p}/\hbar} = \mathbb{I}$$

shows. Because $[\hat{p}, \hat{p}] = 0$ one can calculate these exponents as if they were just numbers. The translation operator satisfies therefore $\hat{T}^\dagger(\alpha) = \hat{T}^{-1}(\alpha) = \hat{T}(-\alpha)$.

The commutator of the translation operator with the position operator is

$$\left[\hat{x}, \hat{T}(\alpha)\right] = \left[\hat{x}, e^{-i\,\alpha\,\hat{p}/\hbar}\right] = [\hat{x}, \hat{p}]\left(-\frac{i\,\alpha}{\hbar}\right) e^{-i\,\alpha\,\hat{p}/\hbar} = i\,\hbar\left(-\frac{i\,\alpha}{\hbar}\right)\hat{T}(\alpha) = \alpha\,\hat{T}(\alpha)$$

using $[\hat{x}, F(\hat{p})] = [\hat{x}, \hat{p}]\,F'(\hat{p}) = i\,\hbar\,F'(\hat{p})$. This gives all that is needed to proof that the $\hat{T}(\alpha)$ is indeed the translation operator.

$\hat{T}(\alpha)$ is a translation operator that translates a ket $|x\rangle$ by an amount of α. It is also comprehensible that $\hat{T}^{-1}(\alpha) = \hat{T}(-\alpha)$. The translation operator is $\hat{T}(-\alpha)\,|x\rangle = |x - \alpha\rangle$ is $\langle x - \alpha| = \langle x|\,\hat{T}^\dagger(-\alpha) = \langle x|\,\hat{T}(\alpha)$ in dual space.

5.6 Hermite Polynomials

The Hermite polynomials $H_n(z)$ are of interest here because the energy eigenfunctions of the quantum harmonic oscillator are written in terms of Hermite polynomials. There are two conventions where one is called the probabilist's convention and the other one is called the physicist's convention. The one used here is the second convention. The definition

$$H_n(z) = (-1)^n \, e^{z^2} \, \frac{d^n}{dz^n} e^{-z^2} \tag{5.7}$$

shows that the Hermite polynomials are defined as the n^{th} derivative of a Gaussian and a prefactor, but it is not immediately clear why this defines polynomials. Looking at the examples

$$H_0(z) = (-1)^0 e^{z^2} e^{-z^2} = 1 \qquad H_1(z) = (-1)^1 e^{z^2}(-2z)e^{-z^2} = 2z \qquad H_2(z) = 4z^2 - 2$$

shows that at least the first three cases are polynomials, and the induction step

$$\begin{aligned}
\frac{d^{n-1}}{dz^{n-1}} e^{-z^2} &= (-1)^{n-1} H_{n-1}(z) e^{-z^2} \\
\frac{d^n}{dz^n} e^{-z^2} &= (-1)^{n-1} \left[\left(\frac{d}{dz} H_{n-1}(z) \right) e^{-z^2} + H_{n-1}(z)(-2z)e^{-z^2} \right] \\
&= (-1)^n e^{-z^2} \left[\left(2z - \frac{d}{dz} \right) H_{n-1}(z) \right] \\
\left(2z - \frac{d}{dz} \right) H_{n-1}(z) &= (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2} = H_n(z)
\end{aligned}$$

proves that $H_n(z)$ is a polynomial if $H_{n-1}(z)$ is one. $H_{n-1}(z)$ is a polynomial of degree $n-1$ and $z\,H_{n-1}(z)$ makes it a polynomial of degree n.

As shown in figure 3 the first Hermite polynomial $H_0(z) = 1$ is just a constant, the second $H_1(z) = 2z$ is a straight line, and the third $H_2(z) = 4z^2 - 2$ is a parabola. Note that the scaling on the vertical axis of

$$H_3(z) = 8z^3 - 12z \qquad H_4(z) = 16z^4 - 48z^2 + 12 \qquad H_5 = 32z^5 - 160z^3 + 120z$$

in the second row of the figure is different for each graph.

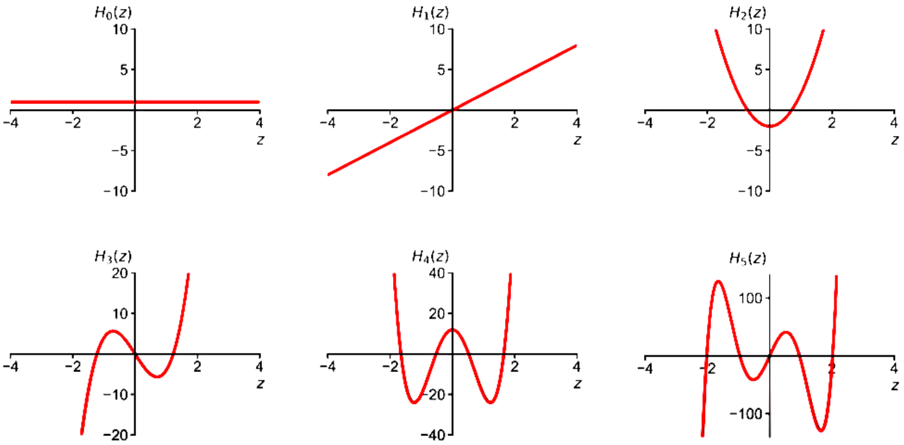


Figure 3: The graphs of the first six Hermite polynomials

All Hermite polynomials have a definite parity as can be shown using induction again. From

$$H_n(z) = \left(2z - \frac{d}{dz} \right) H_{n-1}(z) \quad H_n(-z) = \left(2(-z) - \frac{d}{d(-z)} \right) H_{n-1}(z-) = (-1) \left(2z - \frac{d}{dz} \right) H_{n-1}(-z)$$

one can conclude that $H_n(z)$ is an odd function if $H_{n-1}(z)$ is an even function and vice versa.

Generating functions allow to encode an infinite sequence such as the family of Hermite polynomials by just treating them as the coefficients of a power series. Using $-t^2 + 2tz = z^2 - (z-t)^2$ following from $-(z-t)^2 = -z^2 + 2tz - t^2$, one can write

$$\begin{aligned} g(z,t) = e^{-t^2+2tz} = e^{z^2}\, e^{-(z-t)^2} &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial t^n} g(z,t)\Big|_{t=0} t^n = \sum_{n=0}^{\infty} \frac{1}{n!} \left[(-1)^n e^{z^2} \frac{\partial^n}{\partial z^n} e^{-z^2} \right] t^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} H_n(z)\, t^n = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(z) \end{aligned}$$

because

$$\frac{\partial^n}{\partial t^n} g(z,t)\Big|_{t=0} = e^{z^2} \frac{\partial^n}{\partial t^n} e^{-(z-t)^2}\Big|_{t=0} = (-1)^n e^{z^2} \frac{\partial^n}{\partial z^n} e^{-(z-t)^2}\Big|_{t=0}$$

using $\frac{\partial}{\partial t} f(z-t) = -\frac{\partial}{\partial z} f(z-t)$. With a function $h(x,\lambda)$ defined similarly to $g(z,t)$

$$h(x,\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} \psi_n(x) \qquad\qquad g(z,t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(z)$$

using the eigenfunctions of the harmonic oscillator instead of the Hermite polynomials, one can relate these two functions. Using (5.5) and $\langle x|n\rangle = \psi_n(x)$ one gets

$$h(x,\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} \langle x|n\rangle = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \langle x|(\hat{a}^\dagger)^n|0\rangle = \left\langle x \,\middle|\, \left(\sum_{n=0}^{\infty} \frac{(\lambda \hat{a}^\dagger)^n}{n!} \right) \middle|\, 0 \right\rangle = \langle x| e^{\lambda\, \hat{a}^\dagger} |0\rangle$$

because the infinite sum is the Taylor expansion of an exponential $e^{\lambda\,\hat{a}^\dagger}$.

The exponential $e^{\lambda\,\hat{a}^\dagger}$ can be written using the definition of the raising operator (5.1) in the position representation as

$$e^{\lambda\,\hat{a}^\dagger} = \exp\left[\frac{\lambda}{\sqrt{2}} \left(\sqrt{\frac{m\,\omega}{\hbar}}\, \hat{x} - i \frac{1}{\sqrt{m\,\hbar\,\omega}}\, \hat{p} \right) \right]$$

but the sum e^{a+b} cannot be turned into $e^a\, e^b$ as would be possible for scalars. However, with

$$\hat{C} = \hat{A} + \hat{B} + \frac{1}{2}[\hat{A},\hat{B}] + \frac{1}{12}\Big([\hat{A},[\hat{A},\hat{B}]] - [\hat{B},[\hat{A},\hat{B}]] \Big) + ...$$

called the Baker-Campbell-Hausdorff formula it can be written as $e^{\hat{A}}\, e^{\hat{B}} = e^{\hat{C}}$, and because $[\hat{x},\hat{p}] = i\hbar$ is a scalar only the first commutator remains and all the higher order commutators vanish. Thus,

$$\begin{aligned} e^{\lambda\,\hat{a}^\dagger} &= \exp\left(\sqrt{\frac{m\,\omega}{2\hbar}}\, \lambda\, \hat{x} \right) \exp\left(-i \frac{1}{\sqrt{2m\,\hbar\,\omega}}\, \lambda\, \hat{p} \right) \exp\left(\frac{i\,\lambda^2}{4\hbar} [\hat{x},\hat{p}] \right) \\ &= \exp\left(\sqrt{\frac{m\,\omega}{2\hbar}}\, \lambda\, \hat{x} \right) \exp\left(-i \frac{1}{\sqrt{2m\,\hbar\,\omega}}\, \lambda\, \hat{p} \right) \exp\left(-\frac{\lambda^2}{4} \right) \end{aligned}$$

and using $\langle x|\,\hat{x} = \langle x|\,x$ gives an expression

$$\begin{aligned} h(x,\lambda) &= \exp\left(-\frac{\lambda^2}{4} \right) \langle x| \exp\left(\sqrt{\frac{m\,\omega}{2\hbar}}\, \lambda\, \hat{x} \right) \exp\left(-i \frac{1}{\sqrt{2m\,\hbar\,\omega}}\, \lambda\, \hat{p} \right) |0\rangle \\ &= \exp\left(-\frac{\lambda^2}{4} \right) \exp\left(\sqrt{\frac{m\,\omega}{2\hbar}}\, \lambda\, x \right) \left\langle x \,\middle|\, \exp\left(-i \frac{\lambda}{\sqrt{2m\,\hbar\,\omega}}\, \hat{p} \right) \middle|\, 0 \right\rangle \end{aligned}$$

that looks like the translation operator (5.6) is $\hat{T}(\alpha) = e^{-i\,\alpha\,\hat{p}/\hbar}$. Inserting $\langle x|\,\hat{T}(\alpha) = \langle x - \alpha|$ with the correct factor eliminates the exponent in

$$\langle x| \exp\left(-i \frac{\lambda}{\sqrt{2m\,\hbar\,\omega}}\, \hat{p} \right) = \left\langle x - \sqrt{\frac{\hbar}{2m\,\omega}} \lambda \right|$$

such that one gets

$$h(x,\lambda) = \exp\left(-\frac{\lambda^2}{4}\right)\exp\left(\sqrt{\frac{m\,\omega}{2\hbar}}\,\lambda\,x\right)\left\langle x - \sqrt{\frac{\hbar}{2m\,\omega}}\lambda \,\middle|\, 0\right\rangle$$

and

$$\left\langle x - \sqrt{\frac{\hbar}{2m\,\omega}}\lambda \,\middle|\, 0\right\rangle = \psi_0\left(x - \sqrt{\frac{\hbar}{2m\,\omega}}\lambda\right) \qquad\qquad \psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}$$

with the ground state eigenfunction $\psi_0(x)$. Combining these results gives

$$h(x,\lambda) = \left(\frac{m\,\omega}{\pi\,\hbar}\right)^{1/4}\exp\left(-\frac{\lambda^2}{4}\right)\exp\left(\sqrt{\frac{m\,\omega}{2\hbar}}\,\lambda\,x\right)\exp\left(-\frac{m\,\omega}{2\,\hbar}\left(x - \sqrt{\frac{\hbar}{2m\,\omega}}\lambda\right)^2\right)$$

such that the three exponentials can be combined into

$$-\frac{\lambda^2}{4} + \sqrt{\frac{m\,\omega}{2\hbar}}\,\lambda\,x - \frac{m\,\omega}{2\,\hbar}\left(x^2 - \sqrt{\frac{2\hbar}{m\omega}}\lambda\,x + \frac{\hbar}{2m\omega}\lambda^2\right) = -\frac{m\,\omega}{2\,\hbar}x^2 + 2\sqrt{\frac{m\,\omega}{2\,\hbar}}\lambda\,x - \frac{\lambda^2}{2}$$

$$h(x,\lambda) = \left(\frac{m\,\omega}{\pi\,\hbar}\right)^{1/4}\exp\left(-\frac{m\,\omega}{2\,\hbar}x^2 + 2\sqrt{\frac{m\,\omega}{2\,\hbar}}\lambda\,x - \frac{\lambda^2}{2}\right)$$

because they are scalars. This leads to

$$z = \sqrt{\frac{m\,\omega}{\hbar}}x \qquad t = \frac{\lambda}{\sqrt{2}} \qquad h(z,t) = \left(\frac{m\,\omega}{\pi\,\hbar}\right)^{1/4}\exp\left(-\frac{1}{2}\,z^2 + 2\,t\,z - t^2\right) = \left(\frac{m\,\omega}{\pi\,\hbar}\right)^{1/4} e^{-z^2/2}\,g(z,t)$$

by changing variables and using the generating function $g(z,t) = e^{-t^2+2\,t\,z}$ of the Hermite polynomials introduced above. Combining the infinite sum over the eigenfunctions of the harmonic oscillator and the generating function for the Hermite polynomials

$$h(x,\lambda) = \sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}\psi_n(x) \qquad g(z,t) = \sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(z) \qquad \Rightarrow \qquad \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\lambda}{\sqrt{2}}\right)^2 H_n\left(\sqrt{\frac{m\,\omega}{\hbar}}x\right)$$

finally gives

$$\sum_{n=0}^{\infty}\lambda^n\left(\frac{1}{\sqrt{n!}}\,\psi_n(x)\right) = \left(\frac{m\,\omega}{\pi\,\hbar}\right)^{1/4} e^{-m\,\omega\,x^2/2\hbar}\sum_{n=0}^{\infty}\lambda^n\left[\frac{1}{n!\,\sqrt{2^n}}\,H_n\left(\sqrt{\frac{m\,\omega}{\hbar}}\,x\right)\right]$$

where the terms in the sum must be equal. The energy eigenfunctions of the harmonic oscillator are

$$\psi_n(x) = \frac{1}{\sqrt{2^n\,n!}}\left(\frac{m\,\omega}{\pi\,\hbar}\right)^{1/4} e^{-m\,\omega\,x^2/2\hbar}\,H_n\left(\sqrt{\frac{m\,\omega}{\hbar}}\,x\right) \tag{5.8}$$

in terms of the Hermite polynomials.

5.7 Coherent States of the Quantum Harmonic Oscillator

Coherent states play an important role in quantum mechanics. Two properties are explored where the first one is how they look like in the energy basis and the second is their time evolution. Coherent states of the quantum harmonic oscillator are defined as the eigenstates of the lowering operator. They are the states that most closely resemble the classical motion of a harmonic oscillator. Coherent states (or sometimes called canonical coherent states) are defined as states $|\alpha\rangle$ satisfying

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$$

where $\hat{a}$ is the lowering operator and where $\alpha \in \mathbb{C}$ because $\hat{a}$ is not Hermitian. Given $\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$ it is surprising that $\hat{a}$ can have eigenstates at all.

Writing the coherent state in the basis of energy eigenstates lead to

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n(\alpha)\,|n\rangle \;\text{ with } c_n(\alpha) = \langle\alpha|n\rangle$$

and to

$$\hat{a}\,|\alpha\rangle = \hat{a}\sum_{n=0}^{\infty} c_n(\alpha)\,|n\rangle = \sum_{n=0}^{\infty} c_n(\alpha)\,\hat{a}\,|n\rangle = \sum_{n=1}^{\infty} c_n(\alpha)\,\sqrt{n}\,|n-1\rangle = \sum_{n=0}^{\infty} c_{n+1}(\alpha)\,\sqrt{n+1}\,|n\rangle$$

where the summation index starting at 1 due to $\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$ for $n \geq 1$ and $\hat{a}\,|n\rangle = 0$ for $n = 0$ can be changed to $n+1$. Using the definition of a coherent state $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$ it follows that

$$\sum_{n=0}^{\infty} c_{n+1}(\alpha)\,\sqrt{n+1}\,|n\rangle = \alpha\sum_{n=0}^{\infty} c_n(\alpha)\,|n\rangle \qquad \Rightarrow \qquad \sum_{n=0}^{\infty}\Big[c_{n+1}(\alpha)\,\sqrt{n+1} - \alpha\,c_n(\alpha)\Big]\,|n\rangle = 0$$

and that all the coefficients $c_{n+1}(\alpha)\,\sqrt{n+1} - \alpha\,c_n(\alpha)$ must vanish. Thus,

$$c_{n+1}(\alpha) = \frac{\alpha}{\sqrt{n+1}}\,c_n(\alpha) = \frac{\alpha}{\sqrt{n+1}}\frac{\alpha}{\sqrt{n}}\,c_{n-1}(\alpha) = \ldots \quad c_n(\alpha) = \frac{\alpha}{\sqrt{n}}\frac{\alpha}{\sqrt{n-1}}\ldots\frac{\alpha}{\sqrt{2}}\frac{\alpha}{\sqrt{1}}\,c_0(\alpha) = \frac{\alpha^n}{\sqrt{n!}}\,c_0(\alpha)$$

follows and gives the coherent state

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n(\alpha)\,|n\rangle = \sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,c_0(\alpha)\,|n\rangle$$

in the energy basis. Using normalization

$$1 = \sum_{n=0}^{\infty}|c_n(\alpha)|^2 = \sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!}\,|c_0(\alpha)|^2 = |c_0(\alpha)|^2\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!} = |c_0(\alpha)|^2\,e^{|\alpha|^2}$$

shows that $|c_0(\alpha)|^2 = e^{-|\alpha|^2}$. The conventional phase choice makes $c_0(\alpha)$ real such that $c_0(\alpha) = e^{-|\alpha|^2/2}$, and the coherent state becomes

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle \tag{5.9}$$

as the final expression for the coherent state of the harmonic oscillator in the energy basis.

The resulting $|\alpha\rangle$ is an infinite sum over all energy eigenstates and acting with $\hat{a}$ on it lowers each energy eigenstate by one but keeps the total infinite sum the same. Another interesting point is that the coherent states are associated with the Poisson distribution. The probability for an energy eigenvalue is

$$P(E_n) = |c_n(\alpha)|^2 = \left|e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\right|^2 = \frac{|\alpha|^{2n}}{n!}\,e^{-|\alpha|^2}$$

and this is a Poisson distribution. This property has important implications. For example, coherent states of photons can be split into other independent coherent states in the field of quantum optics, and this can only happen with Poisson statistics.

The time evolution is governed by the Schrödinger equation, and the quantum harmonic oscillator is a conservative system because the Hamiltonian $\hat{H}$ is independent of time t. The time evolution of a coherent state $\psi(t)$ with

$$|\psi(0)\rangle = |\alpha_0\rangle = e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{\alpha_0^n}{\sqrt{n!}}\,|n\rangle$$

at time $t = 0$ is

$$|\psi(t)\rangle = |\alpha_0\rangle = e^{-|\alpha_0|^2/2} \sum_{n=0}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}} e^{-i\,E_n\,t/\hbar}\,|n\rangle = e^{-|\alpha_0|^2/2} \sum_{n=0}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}} e^{-i\,\hbar\,\omega(n+\frac{1}{2})t/\hbar}\,|n\rangle$$
$$= e^{-i\,\omega\,t/2}\, e^{-|\alpha_0|^2/2} \sum_{n=0}^{\infty} \frac{(\alpha_0\, e^{-i\,\omega\,t})^n}{\sqrt{n!}}\,|n\rangle$$

using $E_n = \hbar\,\omega(n + \frac{1}{2})$ from equation (5.4). Setting $\alpha = \alpha_0\, e^{-i\,\omega\,t}$ such that $|\alpha|^2 = |\alpha_0|^2$ the coherent state becomes

$$|\psi(t)\rangle = e^{-i\,\omega\,t/2}\, e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle = e^{-i\,\omega\,t/2}\,|\alpha\rangle$$

for $\alpha = \alpha_0\, e^{-i\,\omega\,t}$ at time t. This means that if the initial state $|\psi(0)\rangle = |\alpha_0\rangle$ is a coherent state then $|\psi(t)\rangle = e^{-i\,\omega\,t/2}\,|\alpha_0\, e^{-i\,\omega\,t}\rangle$ is a global phase factor (that can be ignored since it does not change the physics) times a new coherent state. In other words, a coherent state stays coherent at all times.

Because the coherent states are eigenstates of the lowering operator, a natural question is whether there are eigenstates of the raising operator $\hat{a}^\dagger$. It turns out that the raising operator has no eigenstates. This can be shown starting from the eigenvalue equation $\hat{a}^\dagger\,|\lambda\rangle = \lambda\,|\lambda\rangle$. Inserting the expansion in terms of energy eigenstates gives

$$\hat{a}^\dagger\,|\lambda\rangle = \hat{a}^\dagger \sum_{n=0}^{\infty} c_n\,|n\rangle = \sum_{n=0}^{\infty} c_n\,\hat{a}^\dagger\,|n\rangle = \sum_{n=0}^{\infty} c_n\,\sqrt{n+1}\,|n+1\rangle = \lambda \sum_{n=0}^{\infty} c_n\,|n\rangle$$

and comparing the coefficients shows for $|0\rangle$ that $0 = \lambda c_0$ and therefore $c_0 = 0$ and for $|1\rangle$ that $c_0 = \lambda c_1$ and therefore $c_1 = 0$ and so on. For $|n\rangle$ follows $c_{n-1}\sqrt{n} = \lambda c_n$ and therefore also $c_n = 0$. The key difference between lowering and raising operator for these arguments is that there is a lowest energy eigenstate but no highest energy eigenstate.

5.8 Quasi-Classical States of the Quantum Harmonic Oscillator

Classically, a harmonic oscillator oscillates but the energy eigenstates of a quantum harmonic oscillator are stationary and do not describe anything like an oscillation. Thus, one can ask the question whether the quantum harmonic oscillator actually oscillates. The answer is yes but a coherent state is needed to see those oscillations, and this is why coherent states are also called quasi-classical states.

The classical harmonic oscillator has position, momentum and energy

$$x(t) = x_0\,\cos(\omega t - \varphi) \qquad p(t) = m\,\dot{x}(t) = -m\,\omega\,x_0\,\sin(\omega t - \varphi) \qquad E(t) = \frac{1}{2}\,m\,\omega^2\,x_0^2$$

where the phase φ depends on the initial conditions, and the energy is because of

$$E(t) = \frac{1}{2m}\,[p(t)]^2 + \frac{1}{2}\,m\,\omega^2\,[x(t)]^2 = \frac{1}{2m} m^2\,\omega^2\,x_0^2\,\sin^2(\omega t - \varphi) + \frac{1}{2}\,m\,\omega^2\,x_0^2\cos^2(\omega t - \varphi)$$
$$= \frac{1}{2}\,m\,\omega^2\,x_0^2\left[\sin^2(\omega t - \varphi) + \cos^2(\omega t - \varphi)\right] = \frac{1}{2}\,m\,\omega^2\,x_0^2$$

independent of time. The three quantities $x(t)$, $p(t)$ and $E(t) = E$ fully characterize the classical harmonic oscillator.

Because of Ehrenfest's theorem the classical world and the quantum world can be connected using the expectation values of position and momentum. The expectation value of the position operator in an energy eigenstate $|n\rangle$ can be written as

$$\langle\hat{x}\rangle_n = \langle n|\sqrt{\frac{\hbar}{2\,m\,\omega}}(\hat{a} + \hat{a}^\dagger)|n\rangle = 0$$

using (5.1) and

$$\langle n|\hat{a}|n\rangle = \sqrt{n}\,\langle n|n-1\rangle = 0 \qquad\qquad \langle n|\hat{a}^\dagger|n\rangle = 0$$

because the energy eigenstates build an orthonormal basis. As energy eigenstates are stationary states their expectation value is time independent. With a similar argument one can also show $\langle\hat{p}\rangle_n = 0$. For the Hamiltonian one gets the corresponding eigenvalue $\langle\hat{H}\rangle_n(t) = \hbar\omega(n+\frac{1}{2})$. Thus, the energy $\langle\hat{H}\rangle_n(t)$ and the expectation values of $\langle\hat{x}\rangle_n(t)$ and $\langle\hat{p}\rangle_n(t)$ are all constant for the quantum harmonic oscillator while only the energy $E(t)$ is constant for the classical harmonic oscillator. Thus, energy eigenstates do not show an oscillation that a classical harmonic oscillator exhibits.

Coherent states (5.9) are infinite linear combinations of energy eigenstates. To explore the connection between coherent states and the classical harmonic oscillator (with the reason why coherent states are also called quasi-classical states) one needs to evaluate the expectation values of the position, the momentum and the energy operators in a coherent state. To calculate them some other expectation values have to be determined first.

The eigenvalue equation for the lowering operator is $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$ with $\langle\alpha|\,\hat{a}^\dagger = \langle\alpha|\,\alpha^*$ in the dual space, and the expectation value of $\hat{a}$ in state $|\alpha\rangle$ is $\langle\alpha|\hat{a}|\alpha\rangle = \alpha$ and $\langle\alpha|\hat{a}^n|\alpha\rangle = \alpha^n$ using the eigenvalue equation of the lowering operator. The expectation value of $\hat{a}^\dagger$ in state $|\alpha\rangle$ is $\langle\alpha|\hat{a}^\dagger|\alpha\rangle = \alpha^*$ and $\langle\alpha|(\hat{a}^\dagger)^n|\alpha\rangle = (\alpha^*)^n$ using the dual eigenvalue equation of the lowering operator. The expectation value $\langle\alpha|\hat{a}^\dagger\,\hat{a}|\alpha\rangle = |\alpha|^2$ follows similarly. For the expectation value $\langle\alpha|\hat{a}\,\hat{a}^\dagger|\alpha\rangle = 1 + |\alpha|^2$ the relation $[\hat{a},\hat{a}^\dagger] = 1$ with the consequence $\hat{a}\,\hat{a}^\dagger = 1 + \hat{a}^\dagger\,\hat{a}$ can be used.

The expectation value of the position operator and its square with respect to a coherent state is

$$\begin{aligned}
\langle\hat{x}\rangle_\alpha &= \langle\alpha|\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger)|\alpha\rangle = \sqrt{\frac{\hbar}{2m\omega}}\big(\,\langle\alpha|\hat{a}|\alpha\rangle + \langle\alpha|\hat{a}^\dagger|\alpha\rangle\,\big) = \sqrt{\frac{\hbar}{2m\omega}}(\alpha+\alpha^*) = \sqrt{\frac{\hbar}{m\omega}}\,\mathrm{Re}(\alpha) \\
\langle\hat{x}^2\rangle_\alpha &= \langle\alpha|\frac{\hbar}{2m\omega}(\hat{a}+\hat{a}^\dagger)|\alpha\rangle = \frac{\hbar}{2m\omega}\big(\,\langle\alpha|\hat{a}^2|\alpha\rangle + \langle\alpha|(\hat{a}^\dagger)^2|\alpha\rangle + \langle\alpha|\hat{a}\,\hat{a}^\dagger|\alpha\rangle + \langle\alpha|\hat{a}^\dagger\,\hat{a}|\alpha\rangle\,\big) \\
&= \frac{\hbar}{2m\omega}\Big(\alpha^2 + (\alpha^*)^2 + 2\alpha\alpha^* + 1\Big) = \frac{\hbar}{2m\omega}\Big[\big(\alpha+\alpha^*\big)^2 + 1\Big]
\end{aligned}$$

and the root mean square deviation is

$$\Delta\hat{x}_\alpha = \sqrt{\langle\hat{x}^2\rangle_\alpha + \langle\hat{x}\rangle_\alpha^2} = \sqrt{\frac{\hbar}{2m\omega}\Big[\big(\alpha+\alpha^*\big)^2+1\Big] - \frac{\hbar}{2m\omega}\big(\alpha+\alpha^*\big)^2} = \sqrt{\frac{\hbar}{2m\omega}}$$

but surprisingly does not depend on α.

To determine the time dependence of the expectation value of the position operator, the initial state is assumed to be $|\psi(0)\rangle = |\alpha_0\rangle$ where $\alpha_0 = |\alpha_0|\,e^{i\varphi} \in \mathbb{C}$ and the position is

$$|\psi(t)\rangle = e^{-i\omega t/2}\,|\alpha\rangle \qquad\qquad \alpha = \alpha_0\,e^{-i\omega t} = |\alpha_0|\,e^{-i(\omega t-\varphi)}$$

at time t. The time dependent expectation value of the position operator becomes

$$\langle\hat{x}\rangle_\alpha(t) = \sqrt{\frac{\hbar}{2m\omega}}\Big(|\alpha_0|\,e^{-i(\omega t-\varphi)} + |\alpha_0|\,e^{i(\omega t-\varphi)}\Big) = \sqrt{\frac{2\hbar}{m\omega}}\,|\alpha_0|\,\cos(\omega t-\varphi)$$

using $e^{i\delta} + e^{-i\delta} = 2\cos(\delta)$.

A similar calculation for the expectation value of the momentum operator and its square with the root mean square deviation gives

$$\langle\hat{p}\rangle_\alpha = -i\sqrt{\frac{m\hbar\omega}{2}}(\alpha-\alpha^*) = \sqrt{2m\hbar\omega}\,\mathrm{Im}(\alpha) \qquad \langle\hat{p}^2\rangle_\alpha = \frac{m\hbar\omega}{2}\Big[1-\big(\alpha-\alpha^*\big)^2\Big] \qquad \Delta\hat{p}_\alpha = \sqrt{\frac{m\hbar\omega}{2}}$$

where $\Delta\hat{p}_\alpha$ is also independent of α. With $|\psi(0)\rangle = |\alpha_0\rangle$, $|\psi(t)\rangle = e^{-i\omega t/2}\,|\alpha\rangle$ and $\alpha = |\alpha_0|\,e^{-i(\omega t-\varphi)}$ the time dependence of the expectation value of the momentum operator becomes

$$\langle\hat{p}\rangle_\alpha(t) = -\sqrt{2m\hbar\omega}\,|\alpha_0|\,\sin(\omega t-\varphi)$$

using $e^{i\delta} - e^{-i\delta} = 2i\sin(\delta)$. One gets further

$$\langle \hat{H} \rangle_\alpha = \hbar\omega \left(|\alpha|^2 + \frac{1}{2} \right) \qquad \langle \hat{H}^2 \rangle_\alpha = \hbar^2\omega^2 \left(|\alpha|^4 + 2|\alpha|^2 + \frac{1}{4} \right) \qquad \Delta\hat{H}_\alpha = \hbar\omega|\alpha|$$

$$\langle \hat{H} \rangle_\alpha (t) = \hbar\omega \left(|\alpha_0|^2 + \frac{1}{2} \right)$$

for the Hamiltonian showing that $\langle \hat{H} \rangle_\alpha (t)$ is time independent.

Comparing the classical with this coherent case exhibits the similarities

$$\begin{aligned} x(t) &= x_0 \cos(\omega t - \varphi) & \langle \hat{x} \rangle_\alpha (t) &= \sqrt{\frac{2\hbar}{m\omega}}\, |\alpha_0| \cos(\omega t - \varphi) \\ p(t) &= m\,\dot{x}(t) = -m\,\omega\,x_0 \sin(\omega t - \varphi) & \langle \hat{p} \rangle_\alpha (t) &= -\sqrt{2m\hbar\omega}\, |\alpha_0| \sin(\omega t - \varphi) \\ E(t) &= \frac{1}{2}\, m\,\omega^2\, x_0^2 & \langle \hat{H} \rangle_\alpha (t) &= \hbar\omega \left(|\alpha_0|^2 + \frac{1}{2} \right) \end{aligned}$$

and shows that the quantum particle oscillates when in a coherent state. If one identifies x_0 such that

$$x_0 = \sqrt{\frac{2\hbar}{m\omega}} |\alpha_0| \qquad \Rightarrow \qquad x(t) = \langle \hat{x} \rangle_\alpha (t) \text{ and } p(t) = \langle \hat{p} \rangle_\alpha (t)$$

then the time evolution of position and momentum of the classical motion is exactly reproduced by the quantum motion in the coherent state. This means that if the quantum harmonic oscillator is in a coherent state then the expectation values of position and momentum oscillate in such a way that they exactly reproduce the classical motion. With the same x_0 the energy becomes $E(t) = \langle \hat{H} \rangle_\alpha (t) - \frac{1}{2}\hbar\omega$. Both terms are time independent but they differ by $-\frac{1}{2}\hbar\omega$, and this is a quantum phenomenon. This difference is the zero point energy $E_0 = \frac{1}{2}\hbar\omega$, and since $|\alpha_0| \gg E_0$ for a classical harmonic oscillator the energy can also be seen as equal.

5.9 Displacement Operator

Since one can generate a coherent state using the displacement operator on the ground state of the quantum harmonic oscillator, it plays a crucial role in the study of coherent states. It also plays an important role in other areas such as in quantum optics.

The *displacement operator* is defined as

$$\hat{D}(\alpha) = e^{\alpha\, \hat{a}^\dagger - \alpha^*\, \hat{a}} \tag{5.10}$$

where α is a complex scalar. Using the special case of the Baker-Campbell-Hausdorff formula with

$$\big[\hat{A}, [\hat{A}, \hat{B}]\big] = \big[\hat{B}, [\hat{A}, \hat{B}]\big] = 0 \qquad \Rightarrow \qquad e^{\hat{A}+\hat{B}} = e^{\hat{A}}\, e^{\hat{B}}\, e^{-\frac{1}{2}[\hat{A},\hat{B}]}$$

for $\hat{A} = \hat{a}$, $\hat{B} = \hat{a}^\dagger$ and $[\hat{a}, \hat{a}^\dagger] = 1$ satisfying the preconditions because the commutator is a scalar, the displacement operator can be written as

$$\hat{D}(\alpha) = e^{\alpha\, \hat{a}^\dagger}\, e^{-\alpha^* \hat{a}}\, e^{-\frac{1}{2}[\alpha\, \hat{a}^\dagger, -\alpha^* \hat{a}]} = e^{\alpha\, \hat{a}^\dagger}\, e^{-\alpha^* \hat{a}}\, e^{-|\alpha|^2/2} = e^{-|\alpha|^2/2}\, e^{\alpha\, \hat{a}^\dagger}\, e^{-\alpha^* \hat{a}}$$

because $[\alpha\, \hat{a}^\dagger, -\alpha^* \hat{a}] = -\alpha^*\alpha[\hat{a}^\dagger, \hat{a}] = -|\alpha|^2(-1) = |\alpha|^2$.

The adjoint of the displacement operator is

$$\hat{D}^\dagger(\alpha) = e^{\alpha^*\, \hat{a} - \alpha\, \hat{a}^\dagger} = e^{|\alpha|^2/2}\, e^{\alpha^* \hat{a}}\, e^{-\alpha\, \hat{a}^\dagger}$$

using $(e^{\hat{A}})^\dagger = e^{\hat{A}^\dagger}$. This shows $\hat{D}(\alpha)\, \hat{D}^\dagger(\alpha) = \hat{D}^\dagger(\alpha)\, \hat{D}(\alpha) = \mathbb{I}$ because the exponents of the corresponding exponentials commute. Thus, $\hat{D}^\dagger(\alpha) = \hat{D}^{-1}(\alpha)$ and the displacement operator is unitary. Again using the commutation relations proves $\hat{D}(-\alpha) = \hat{D}^\dagger(\alpha)$ such that

$$\hat{D}^\dagger(\alpha) = \hat{D}^{-1}(\alpha) = \hat{D}(-\alpha)$$

follows. The next property is

$$\hat{D}(\alpha)\,\hat{D}(\beta) = e^{(\alpha\beta^* - \alpha^*\beta)/2}\,\hat{D}(\alpha+\beta)$$

and can be proven

$$\begin{aligned}\hat{D}(\alpha+\beta) &= e^{\alpha\,\hat{a}^\dagger - \alpha^*\,\hat{a} + \beta\,\hat{a}^\dagger - \beta^*\,\hat{a}} = e^{\alpha\,\hat{a}^\dagger - \alpha^*\,\hat{a}}\,e^{\beta\,\hat{a}^\dagger - \beta^*\,\hat{a}}\,e^{-\frac{1}{2}[\alpha\,\hat{a}^\dagger - \alpha^*\,\hat{a}, \beta\,\hat{a}^\dagger - \beta^*\,\hat{a}]} \\ &= \hat{D}(\alpha)\,\hat{D}(\beta)\,e^{-\frac{1}{2}[\alpha\,\hat{a}^\dagger - \alpha^*\,\hat{a}, \beta\,\hat{a}^\dagger - \beta^*\,\hat{a}]} = \hat{D}(\alpha)\,\hat{D}(\beta)\,e^{-\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)}\end{aligned}$$

using the same version of the Baker-Campbell-Hausdorff formula as above and the derivation

$$[\alpha\,\hat{a}^\dagger - \alpha^*\,\hat{a}, \beta\,\hat{a}^\dagger - \beta^*\,\hat{a}] = -\alpha\beta^*[\hat{a}^\dagger, \hat{a}] - \alpha^*\beta[\hat{a}, \hat{a}^\dagger] = -\alpha\beta^*(-1) - \alpha^*\beta(+1) = \alpha\beta^* - \alpha^*\beta$$

for the scalar exponent. This property is the key to understand the name displacement operator.

Two commutation relations are needed, and the first one is

$$\begin{aligned}\left[\hat{a}, \hat{D}(\alpha)\right] &= \left[\hat{a}, e^{-|\alpha|^2/2}\,e^{\alpha\,\hat{a}^\dagger}\,e^{-\alpha^*\hat{a}}\right] = e^{-|\alpha|^2/2}\left[\hat{a}, e^{\alpha\,\hat{a}^\dagger}\,e^{-\alpha^*\hat{a}}\right] \\ &= e^{-|\alpha|^2/2}\left(e^{\alpha\,\hat{a}^\dagger}\left[\hat{a}, e^{-\alpha^*\hat{a}}\right] + \left[\hat{a}, e^{\alpha\,\hat{a}^\dagger}\right]e^{-\alpha^*\hat{a}}\right) = e^{-|\alpha|^2/2}\left[\hat{a}, e^{\alpha\,\hat{a}^\dagger}\right]e^{-\alpha^*\hat{a}} \\ &= e^{-|\alpha|^2/2}[\hat{a}, \hat{a}^\dagger]\alpha\,e^{\alpha\hat{a}^\dagger}\,e^{-\alpha^*\hat{a}} = \alpha\,e^{-|\alpha|^2/2}\,e^{\alpha\hat{a}^\dagger}\,e^{-\alpha^*\hat{a}} = \alpha\,\hat{D}(\alpha)\end{aligned}$$

using $[\hat{A}, \hat{B}\hat{C}] = \hat{B}[\hat{A}, \hat{C}] + [\hat{A}, \hat{B}]\hat{C}$ and $[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0 \Rightarrow [\hat{A}, F(\hat{B})] = [\hat{A}, \hat{B}]F'(\hat{B})$ as well as $[\hat{a}, e^{-\alpha^*\hat{a}}] = 0$. A similar calculation gives the second commutation relation and

$$\left[\hat{a}, \hat{D}(\alpha)\right] = \alpha\,\hat{D}(\alpha) \qquad\qquad \left[\hat{a}^\dagger, \hat{D}(\alpha)\right] = \alpha^*\,\hat{D}(\alpha)$$

lists both of them.

An important property of unitary operators is that they conserve the norm of quantum states. This property is important for translation in space and time evolution. The unitary transformation of $\hat{a}$ and $\hat{a}^\dagger$ with respect to the displacement operator is

$$\begin{aligned}&\hat{D}(\alpha)^\dagger\,\hat{a}\,\hat{D}(\alpha) = \hat{D}(\alpha)^\dagger\left(\hat{D}(\alpha)\,\hat{a} + \alpha\,\hat{D}(\alpha)\right) = \hat{D}(\alpha)^\dagger\,\hat{D}(\alpha)\,\hat{a} + \hat{D}(\alpha)^\dagger\,\hat{D}(\alpha)\,\alpha = \mathbb{I}\,\hat{a} + \mathbb{I}\,\alpha = \hat{a} + \alpha \\ &\hat{D}(\alpha)^\dagger\,\hat{a}^\dagger\,\hat{D}(\alpha) = \hat{a}^\dagger + \alpha^*\end{aligned}$$

because of $\hat{a}\,\hat{D}(\alpha) = \hat{D}(\alpha)\,\hat{a} + \alpha\,\hat{D}(\alpha)$ following from the first commutator and a similar derivation for the second one.

The displacement operator can be applied in the context of coherent states $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$ because

$$|\alpha\rangle = \hat{D}(\alpha)\,|0\rangle \tag{5.11}$$

or, in words, because one can write a coherent state $|\alpha\rangle$ as equal to the displacement operator applied to the ground state of the quantum harmonic oscillator. To prove this one shows

$$\hat{a}\left(\hat{D}(\alpha)\,|0\rangle\right) = \hat{a}\,\hat{D}(\alpha)\,|0\rangle = \left(\hat{D}(\alpha)\,\hat{a} + \alpha\,\hat{D}(\alpha)\right)|0\rangle = \hat{D}(\alpha)\,\hat{a}\,|0\rangle + \alpha\,\hat{D}(\alpha)\,|0\rangle = 0 + \alpha\left(\hat{D}(\alpha)\,|0\rangle\right)$$

again using $\hat{a}\,\hat{D}(\alpha) = \hat{D}(\alpha)\,\hat{a} + \alpha\,\hat{D}(\alpha)$. Therefore, $\hat{D}(\alpha)\,|0\rangle$ is an eigenstate of $\hat{a}$ with eigenvalue α, and one can generate the coherent state $|\alpha\rangle$ by the application of the displacement operator on the ground state. This is, by the way, an alternative but equivalent definition of a coherent state.

The application of $\hat{D}(\beta)$ to the state $|\alpha\rangle = \hat{D}(\alpha)\,|0\rangle$ gives

$$\begin{aligned}\hat{D}(\beta)\,|\alpha\rangle &= \hat{D}(\beta)\,\hat{D}(\alpha)\,|0\rangle \\ &= e^{(\alpha\beta^* - \alpha^*\beta)/2}\,\hat{D}(\alpha+\beta)\,|0\rangle = e^{(\alpha\beta^* - \alpha^*\beta)/2}\,|\alpha+\beta\rangle\end{aligned}$$

and this is an irrelevant factor times another coherent state that is displaced by β compared to the original one. This is obviously the reason why this operator is called the displacement operator.

5.10 Coherent State Wave Function

In the position representation with $\hat{x}\,|x\rangle = x\,|x\rangle$ a state $|\psi\rangle$ is defined as

$$|\psi\rangle = \int dx\,\psi(x)\,|x\rangle \qquad\qquad \psi(x) = \langle x|\psi\rangle$$

and $\psi(x)$ is called the wave function. The wave functions of coherent states is defined as $\psi_\alpha(x) = \langle x|\alpha\rangle$ where $|\alpha\rangle$ is the coherent state. The coherent state wave function is therefore

$$\psi_\alpha(x) = \langle x|\alpha\rangle = \langle x|\hat{D}(\alpha)|0\rangle$$

using (5.11). The exponent in the definition (5.10) can be written as

$$\begin{aligned}
\alpha\,\hat{a}^\dagger - \alpha^*\,\hat{a} &= \alpha\left(\sqrt{\frac{m\,\omega}{2\hbar}}\,\hat{x} - i\frac{1}{\sqrt{2m\,\hbar\,\omega}}\,\hat{p}\right) - \alpha^*\left(\sqrt{\frac{m\,\omega}{2\hbar}}\,\hat{x} + i\frac{1}{\sqrt{2m\,\hbar\,\omega}}\,\hat{p}\right)\\
&= \sqrt{\frac{m\,\omega}{2\hbar}}(\alpha-\alpha^*)\hat{x} - \frac{i}{\sqrt{2m\,\hbar\,\omega}}(\alpha+\alpha^*)\hat{p}
\end{aligned}$$

in terms of the position and momentum operators with (5.1). The displacement operator can therefore be rewritten as

$$\begin{aligned}
\hat{D}(\alpha) &= \exp\left(\sqrt{\frac{m\,\omega}{2\hbar}}(\alpha-\alpha^*)\hat{x} - \frac{i}{\sqrt{2m\,\hbar\,\omega}}(\alpha+\alpha^*)\hat{p}\right)\\
&= \exp\left(\sqrt{\frac{m\,\omega}{2\hbar}}(\alpha-\alpha^*)\hat{x}\right)\cdot\exp\left(-\frac{i}{\sqrt{2m\,\hbar\,\omega}}(\alpha+\alpha^*)\hat{p}\right)\\
&\quad\cdot\exp\left(-\frac{1}{2}\left[\sqrt{\frac{m\,\omega}{2\hbar}}(\alpha-\alpha^*)\hat{x}, -\frac{i}{\sqrt{2m\,\hbar\,\omega}}(\alpha+\alpha^*)\hat{p}\right]\right)\\
&= \exp\left(\sqrt{\frac{m\,\omega}{2\hbar}}(\alpha-\alpha^*)\hat{x}\right)\cdot\exp\left(-\frac{i}{\sqrt{2m\,\hbar\,\omega}}(\alpha+\alpha^*)\hat{p}\right)\cdot\exp\left(-\frac{1}{4}(\alpha^2-(\alpha^*)^2)\right)
\end{aligned}$$

using the Baker-Campbell-Hausdorff formula for $[\hat{x},\hat{p}] = i\hbar$.

Inserting this result for the displacement operator into $\psi_\alpha(x) = \langle x|\hat{D}(\alpha)|0\rangle$ gives in terms of the ground-state wave function $\psi_0(x)$

$$\begin{aligned}
\psi_\alpha(x) &= \exp\left(-\frac{1}{4}(\alpha^2-(\alpha^*)^2)\right)\left\langle x\left|\exp\left(\sqrt{\frac{m\,\omega}{2\hbar}}(\alpha-\alpha^*)\hat{x}\right)\,\exp\left(-\frac{i}{\sqrt{2m\,\hbar\,\omega}}(\alpha+\alpha^*)\hat{p}\right)\right|0\right\rangle\\
&= \exp\left(-\frac{1}{4}(\alpha^2-(\alpha^*)^2)\right)\exp\left(\sqrt{\frac{m\,\omega}{2\hbar}}(\alpha-\alpha^*)x\right)\left\langle x\left|\exp\left(-\frac{i}{\sqrt{2m\,\hbar\,\omega}}(\alpha+\alpha^*)\hat{p}\right)\right|0\right\rangle\\
&= \exp\left(-\frac{1}{4}(\alpha^2-(\alpha^*)^2)\right)\exp\left(\sqrt{\frac{m\,\omega}{2\hbar}}(\alpha-\alpha^*)x\right)\left\langle x\left|\hat{T}\left(\sqrt{\frac{\hbar}{2m\omega}}(\alpha+\alpha^*)\right)\right|0\right\rangle\\
&= \exp\left(-\frac{1}{4}(\alpha^2-(\alpha^*)^2)\right)\exp\left(\sqrt{\frac{m\,\omega}{2\hbar}}(\alpha-\alpha^*)x\right)\left\langle x - \sqrt{\frac{\hbar}{2m\omega}}(\alpha+\alpha^*)\right|\left.0\right\rangle\\
&= \exp\left(-\frac{1}{4}(\alpha^2-(\alpha^*)^2)\right)\exp\left(\sqrt{\frac{m\,\omega}{2\hbar}}(\alpha-\alpha^*)x\right)\psi_0\left(x - \sqrt{\frac{\hbar}{2m\omega}}(\alpha+\alpha^*)\right)
\end{aligned}$$

by using $\langle x|\,\hat{x} = \langle x|\,x$ and the translation operator (5.6). With

$$\langle\hat{x}\rangle_\alpha = \sqrt{\frac{\hbar}{2m\omega}}(\alpha+\alpha^*) \qquad\qquad \langle\hat{p}\rangle_\alpha = -i\sqrt{\frac{m\hbar\omega}{2}}(\alpha-\alpha^*)$$

the coherent state wave function becomes

$$\psi_\alpha(x) = \exp\left(-\frac{1}{4}(\alpha^2-(\alpha^*)^2)\right)\exp\left(\frac{i}{\hbar}\,\langle\hat{p}\rangle_\alpha\,x\right)\psi_0\big(x - \langle\hat{x}\rangle_\alpha\big)$$

showing that this is a simple shift of the ground state. Using further

$$-\frac{1}{4}\big(\alpha^2 - (\alpha^*)^2 = -i\ \mathrm{Re}(\alpha)\ \mathrm{Im}(\alpha) = i\,\vartheta_\alpha \qquad\qquad \vartheta_\alpha = -\,\mathrm{Re}(\alpha)\ \mathrm{Im}(\alpha) \in \mathbb{R}$$

leads to

$$\psi_\alpha(x) = e^{i\vartheta_\alpha}\, e^{i\langle\hat{p}\rangle_\alpha x/\hbar}\, \psi_0\big(x - \langle\hat{x}\rangle_\alpha\,\big) \tag{5.12}$$

for the wave function of the coherent state $|\alpha\rangle$.

5.11 Minimum Uncertainty States

The states that minimize the uncertainty due to Heisenberg's uncertainty principle are called *minimum uncertainty states*. For position and momentum the uncertainty is

$$\Delta\hat{x}\,\Delta\hat{p} \geq \frac{\hbar}{2}$$

due to $[\hat{x},\hat{p}] = i\hbar$, and the question is what states of the system lead to the condition $\Delta\hat{x}\,\Delta\hat{p} = \frac{\hbar}{2}$.

The derivation here is done again but differently than for the proof of the general uncertainty principle. For a $|\psi\rangle$ the state $|\varphi\rangle = (\hat{\sigma}_x + i\lambda\,\hat{\sigma}_p)\,|\psi\rangle$ is defined where $\lambda\in\mathbb{R}$ and where the two mean square deviations $\hat{\sigma}_x$ and $\hat{\sigma}_p$ are calculated with respect to $|\psi\rangle$. Calculating the scalar product gives

$$\begin{aligned}
\langle\varphi|\varphi\rangle &= \langle\psi|(\hat{\sigma}_x - i\lambda\,\hat{\sigma}_p)(\hat{\sigma}_x + i\lambda\,\hat{\sigma}_p)|\psi\rangle = \langle\psi|\hat{\sigma}_x^2|\psi\rangle + i\lambda\,\langle\psi|(\hat{\sigma}_x\hat{\sigma}_p - \hat{\sigma}_p\hat{\sigma}_x)|\psi\rangle + \lambda^2\,\langle\psi|\hat{\sigma}_p^2|\psi\rangle \\
&= \langle\psi|\hat{\sigma}_x^2|\psi\rangle + i\lambda\,\langle\psi|[\hat{\sigma}_x,\hat{\sigma}_p]|\psi\rangle + \lambda^2\,\langle\psi|\hat{\sigma}_p^2|\psi\rangle = \langle\psi|\hat{\sigma}_x^2|\psi\rangle + i\lambda\,\langle\psi|(i\hbar)|\psi\rangle + \lambda^2\,\langle\psi|\hat{\sigma}_p^2|\psi\rangle \\
&= \langle\hat{\sigma}_x^2\rangle_\psi - \lambda\hbar + \lambda^2\,\langle\hat{\sigma}_p^2\rangle_\psi \geq 0
\end{aligned}$$

using $[\hat{\sigma}_x,\hat{\sigma}_p] = [\hat{x} - \langle\hat{x}\rangle_\psi\,, \hat{p} - \langle\hat{p}\rangle_\psi] = [\hat{x},\hat{p}] = i\hbar$.

As a reminder on quadratic function $f(x) = ax^2 + bx + c$ this function satisfies $f(x)\geq 0$ for all values x if $a>0$ and the function has one or zero roots $f(x)=0$ but not two roots. The well-known formula for the roots is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad\qquad \Delta = b^2 - 4ac$$

where the discriminant Δ determines the number of real roots. The conditions for $f(x)\geq 0$ for all x is therefore $a>0$ and $\Delta\leq 0$ or $4ac\geq b^2$. This means for the above function

$$a = \langle\hat{\sigma}_p^2\rangle_\psi \qquad b = -\hbar \qquad c = \langle\hat{\sigma}_x^2\rangle_\psi \qquad \Rightarrow \qquad \langle\hat{\sigma}_x^2\rangle_\psi\ \langle\hat{\sigma}_p^2\rangle_\psi \geq \frac{\hbar^2}{4}$$

and therefore the Heisenberg uncertainty principle $\Delta\hat{x}\,\Delta\hat{p}\geq\frac{\hbar}{2}$ because of $\Delta\hat{A} = \sqrt{\langle\hat{\sigma}_A^2\rangle_\psi}$.

This derivation of the uncertainty principle is useful because

$$\begin{aligned}
\langle\varphi|\varphi\rangle \geq 0 \quad &\Rightarrow \quad \langle\hat{\sigma}_x^2\rangle_\psi - \lambda\hbar + \lambda^2\,\langle\hat{\sigma}_p^2\rangle_\psi \geq 0 \quad \Rightarrow \quad \Delta\hat{x}\,\Delta\hat{p} \geq \frac{\hbar}{2} \\
\langle\varphi|\varphi\rangle = 0 \quad &\Rightarrow \quad \langle\hat{\sigma}_x^2\rangle_\psi - \lambda\hbar + \lambda^2\,\langle\hat{\sigma}_p^2\rangle_\psi = 0 \quad \Rightarrow \quad \Delta\hat{x}\,\Delta\hat{p} = \frac{\hbar}{2} \\
\langle\varphi|\varphi\rangle = 0 \quad &\Rightarrow \quad |\varphi\rangle = 0 \quad \Rightarrow \quad (\hat{\sigma}_x + i\lambda\,\hat{\sigma}_p)\,|\psi\rangle = 0
\end{aligned}$$

shows the condition for minimal uncertainty states. This gives $((\hat{x} - \langle\hat{x}\rangle_\psi) + i\lambda(\hat{p} - \langle\hat{p}\rangle_\psi))\,|\psi\rangle = 0$ because of $\hat{\sigma}_A = \hat{A} - \langle\hat{A}\rangle_\psi$. The remaining task is to determine $|\psi\rangle$ satisfying this equation in the position representation in terms of wave functions.

The determined condition is

$$\left(x - \langle\hat{x}\rangle_\psi + i\lambda\left(-i\hbar\frac{d}{dx}\right) - \langle\hat{p}\rangle_\psi\right)\psi(x) = 0$$

in position representation, and this is a differential equation. With the not so intuitive change of variable

$$\psi(x) = e^{i\,\langle\hat{p}\rangle_\psi\, x/\hbar}\, u(x - \langle\hat{x}\rangle_\psi)$$

the derivative with respect to x becomes

$$\begin{aligned}\frac{d}{dx}\psi(x) &= \frac{i\,\langle\hat{p}\rangle_\psi}{\hbar}\, e^{i\,\langle\hat{p}\rangle_\psi\, x/\hbar}\, u(x - \langle\hat{x}\rangle_\psi) + e^{i\,\langle\hat{p}\rangle_\psi\, x/\hbar}\,\frac{d}{dx}u(x - \langle\hat{x}\rangle_\psi)\\ &= e^{i\,\langle\hat{p}\rangle_\psi\, x/\hbar}\left(\frac{i\,\langle\hat{p}\rangle_\psi}{\hbar} + \frac{d}{dx}\right)u(x - \langle\hat{x}\rangle_\psi)\end{aligned}$$

giving the differential equations

$$\begin{aligned}e^{i\,\langle\hat{p}\rangle_\psi\, x/\hbar}\left(x - \langle\hat{x}\rangle_\psi + \lambda\hbar\left(\frac{i\,\langle\hat{p}\rangle_\psi}{\hbar} + \frac{d}{dx}\right) - i\lambda\,\langle\hat{p}\rangle_\psi\right)u(x - \langle\hat{x}\rangle_\psi) &= 0\\ \left(\left(x - \langle\hat{x}\rangle_\psi\right) + \lambda\hbar\frac{d}{d(x - \langle\hat{x}\rangle_\psi)}\right)u(x - \langle\hat{x}\rangle_\psi) &= 0\end{aligned}$$

and

$$\left(q + \lambda\hbar\frac{d}{dq}\right)u(q) = 0$$

after substituting $q = x - \langle\hat{x}\rangle_\psi$ and using the fact that $\langle\hat{x}\rangle_\psi$ is a scalar. This differential equation is easy to solve

$$\begin{aligned}\int \frac{1}{u(q)}\frac{du(q)}{dq}\,dq &= -\frac{1}{\lambda\hbar}\int q\,dq\\ \ln\big(u(q)\big) &= -\frac{1}{2\lambda\hbar}q^2 + c \qquad\qquad \Rightarrow \qquad\qquad u(q) = A\,e^{-q^2/2\lambda\hbar}\end{aligned}$$

by using separation of variables and integration. The result is

$$\psi(x) = A\,e^{i\,\langle\hat{p}\rangle_\psi\, x/\hbar}\,e^{-(x-\langle\hat{x}\rangle_\psi)^2/2\lambda\hbar}$$

for the original wave function $\psi(x)$. To determine $\psi(x)$, this wave function is assumed to be normalized such that

$$1 = \int_{-\infty}^{+\infty}|\psi(x)|^2\,dx = |A|^2\int_{-\infty}^{+\infty}e^{-(x-\langle\hat{x}\rangle_\psi)^2/\lambda\hbar}\,dx = |A|^2\sqrt{\pi\lambda\hbar} \qquad \int_{-\infty}^{+\infty}e^{-a(x+b)^2}\,dx = \sqrt{\frac{\pi}{a}}$$

using the Gaussian integral on the right side and the fact that the other exponential is just a phase.

The minimum uncertainty wave function is therefore

$$\psi(x) = \left(\frac{1}{\pi\lambda\hbar}\right)^{1/4} e^{i\,\langle\hat{p}\rangle_\psi\, x/\hbar}\,e^{-(x-\langle\hat{x}\rangle_\psi)^2/2\lambda\hbar}$$

selecting A real and positive with a phase choice. Therefore,

$$|\psi(x)|^2 = \frac{1}{\sqrt{\pi\lambda\hbar}}\,e^{-(x-\langle\hat{x}\rangle_\psi)^2/\lambda\hbar}$$

is a Gaussian. Thus, minimum uncertainty states have Gaussian wave functions and they have already come up in various place such as the ground state of the quantum harmonic oscillator and the coherent states.

Going back to the discriminant $b^2 - 4ac$ where $f(x) = 0$ for $a > 0$ and $b^2 = 4ac$ shows that this is the case for $x_0 = -\frac{b}{2a}$. The corresponding λ_0 is

$$\lambda_0 = \frac{\hbar}{2\,\langle\hat{\sigma}_p^2\rangle_\psi} = \frac{\hbar}{2(\Delta\hat{p})^2} = \frac{2(\Delta\hat{x})^2}{\hbar}$$

using $\langle\hat{\sigma}_p^2\rangle_\psi = (\Delta\hat{p})^2$ and $\Delta\hat{x}\,\Delta\hat{p} = \frac{\hbar}{2}$. The final expression is therefore

$$\psi(x) = \left(\frac{1}{2\pi(\Delta\hat{x})^2}\right)^{1/4} e^{i\,\langle\hat{p}\rangle_\psi\, x/\hbar}\,e^{-\left(\frac{x-\langle\hat{x}\rangle_\psi}{2\Delta\hat{x}}\right)^2} \qquad |\psi(x)|^2 = \frac{1}{\sqrt{2\pi}\Delta\hat{x}}\,e^{-\frac{1}{2}\left(\frac{x-\langle\hat{x}\rangle_\psi}{\Delta\hat{x}}\right)^2} \tag{5.13}$$

for the minimum uncertainty wave function.

5.12 Minimum Uncertainty State and Coherent State Wave Function

The coherent state wave function in (5.12) has certain properties. One of them is

$$|\psi_\alpha(x)|^2 = \left|\psi_0\big(x - \langle\hat{x}\rangle_\alpha\big)\right|^2 = \sqrt{\frac{m\omega}{\pi\hbar}}\exp\left(-\frac{m\omega}{\hbar}\big(x - \langle\hat{x}\rangle_\alpha\big)^2\right)$$
$$= \frac{1}{\sqrt{2\pi}\Delta\hat{x}_\alpha}\exp\left(-\frac{1}{2}\left(\frac{x - \langle\hat{x}\rangle_\alpha}{\Delta\hat{x}_\alpha}\right)^2\right)$$

because the two exponentials are just phases. This is a Gaussian centered at the expectation value of the position operator $\langle\hat{x}\rangle_\alpha$. Using

$$\langle\hat{x}\rangle_\alpha = \sqrt{\frac{\hbar}{2m\omega}} \qquad \langle\hat{p}\rangle_\alpha = \sqrt{\frac{m\hbar\omega}{2}} \qquad \Rightarrow \qquad \langle\hat{x}\rangle_\alpha \langle\hat{p}\rangle_\alpha = \frac{\hbar}{2}$$

for a coherent state shows that coherent states are minimum uncertainty states.

A coherent state $|\alpha\rangle$ can be written as $\hat{D}(\alpha)|0\rangle$. The Gaussian $|\psi_0(x)|^2$ at $x = 0$ becomes the Gaussian $|\psi_\alpha(x)|^2$ at $x = \langle\hat{x}\rangle_\alpha$. Thus, a coherent state is simply a displacement of a Gaussian ground state wave function from the origin to the new position, and this displacement is provided by the action of the displacement operator acting on the ground state.

If $|\psi(0)\rangle = |\alpha_0\rangle$ is a coherent state then $|\psi(t)\rangle = e^{-i\omega t/2}|\alpha\rangle$ at a later time t. A coherent state stays coherent at all times. The time dependence of $|\psi_\alpha(x,t)|^2$ is

$$|\psi_\alpha(x,t)|^2 = |\psi_0(x - \langle\hat{x}\rangle_\alpha(t))|^2 = \frac{1}{\sqrt{2\pi}\Delta\hat{x}_\alpha}\exp\left(-\frac{1}{2}\left(\frac{x - \langle\hat{x}\rangle_\alpha(t)}{\Delta\hat{x}_\alpha}\right)^2\right)$$

with

$$\langle\hat{x}\rangle_\alpha(t) = \sqrt{\frac{2\hbar}{m\omega}}\,|\alpha_0|\cos(\omega t - \varphi) \qquad\qquad \Delta\hat{x}_\alpha = \sqrt{\frac{\hbar}{2m\omega}}$$

where $\langle\hat{x}\rangle_\alpha(t)$ depends on time but $\Delta\hat{x}_\alpha$ is constant.

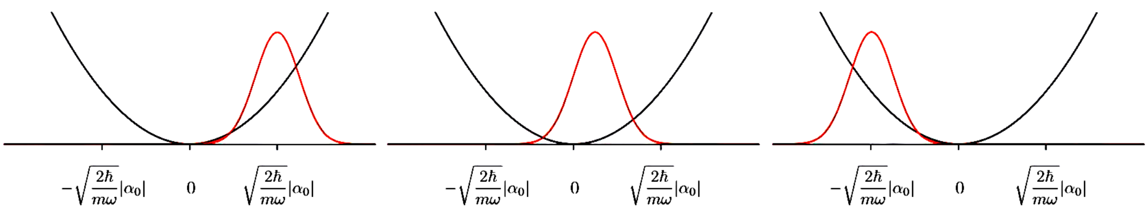


Figure 4: Time evolution of a coherent state

The evolution in time of a coherent state is visualized in figure 4 where the situation at three different times t is shown. The wave packet is a coherent state with the initial position on one extreme of the motion as in this figure on the left side. Time evolution starts, and the packet moves to the left as in the figure in the middle until it reaches the other extreme as in the figure on the right side. Because $\Delta\hat{x}_\alpha$ is independent of time the Gaussian representing the wave packet does not change its width and just moves back and forth where the center moves according to the formula for $\langle\hat{x}\rangle_\alpha(t)$ with $\cos(\omega t - \varphi)$. In other words, the wave package does not spread in time and moves according to a cosine function as the classical harmonic oscillator does.